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The searchlight problem in radiative transfer with internal reflection

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Abstract

We extend the range of the searchlight problem in radiative transfer to include internal reflection arising from Fresnel and Lambert processes. For an isotropic beam and a normal beam, we calculate the albedo, the surface intensity and the mean distance of travel of a photon in the lateral direction. Numerical and graphical results are presented for the above quantities.

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1. Introduction

A classic problem in transport theory is that of a pencil beam of radiation (or neutrons) incident on a half-space at a point on its surface. The seminal work on this problem is due to Elliott (1952, 1955) who considered the closely related problem of a point source on the surface of a half-space. There followed the work of Rybicki (1971), Williams (1982), Siewert and Dunn (1982, 1983), Siewert (1989), Dunn (1985) and Siewert (1982, 1984, 1989). Other relevant references may be found in Siewert who also deals with a slab rather than a half-space. A more recent contribution to the literature is Barichello and Siewert (2000). A series of papers, apparently unknown to those working in neutron transport, was published by Crosbie and co-workers (Breig and Crosbie 1973, Crosbie 1978, Crosbie and Dougherty 1978, Crosbie 1979, Crosbie and Dougherty 1988, Crosbie and Shieh 1990, Crosbie and Shieh 1991, Crosbie and Shieh 1993). This work is of considerable importance and is based upon the integral form of the transport equation. Many of the results obtained by neutron transport workers are derived and it appears that the two groups worked quite independently of each other. We shall refer in more detail to the work of Crosbie *et al* below.

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The present work is an extension of the paper by the author (Williams 1982) to include the effect of internal reflection at the surface. Two cases are considered: specular reflection, governed by the Fresnel laws, and diffuse reflection described by Lambert's law (Born and Wolf 1999). Thus we have a mono-directional beam incident on the surface at the origin and obtain expressions for the albedo, the emergent angular distribution and the surface intensity for various cases. We also obtain a measure of the mean distance of travel in the lateral direction. The procedure we adopt is based on infinite Fourier transforms in the lateral (x–y) directions, a Laplace transform in the normal (z) direction and the principle of superposition. Various schemes are used for inverting the transforms and for obtaining asymptotic estimates. There are some practical uses of the work to be described; namely, to understand better the transport of radiation in tissue in connection with optical tomography in the infra-red region, the dispersal of a laser beam in a medium and the emissive power of an industrial surface subjected to a localized radiative heat source. In addition, we can examine the effect of different types of surface, e.g. specular or diffuse, on the albedo and spatial intensity of radiation in an optical medium.

2. General theory

We denote the angular radiation intensity (neutrons or radiation) by the symbol $I(x, y, z, \mu, \varphi)$, where $\mu = \cos\vartheta$ and φ denote the direction of motion of the photon or neutron with respect to the x, y, z axes (Williams 1971). Thus with isotropic scattering, we may write the transport equation as

$$\left(\mu\frac{\partial}{\partial z} + \sqrt{1-\mu^2}\left\{\cos\varphi\frac{\partial}{\partial x} + \sin\varphi\frac{\partial}{\partial y}\right\} + 1\right) I(x, y, z, \mu, \varphi) = \frac{c}{4\pi}\int_0^{2\pi} d\varphi' \int_{-1}^{1} d\mu' I(x, y, z, \mu', \varphi') \equiv \frac{c}{4\pi} I_0(x, y, z) \tag{1}$$

where $I_0(x, y, z)$ is the scalar intensity, $c = \Sigma_s/(\Sigma_s + \Sigma_a)$ and distance is measured in units of the mean free path. Σ_s and Σ_a are the macroscopic scattering and absorption cross sections, respectively, defined by $\Sigma_s = N_s\sigma_s$ and $\Sigma_a = N_a\sigma_a$, where N_s and N_a are the number densities of the scatterers and absorbers and σ_s and σ_a the microscopic cross sections. Associated with equation (1) is the boundary condition, which we may write as follows for specular reflection

$$I(x, y, 0, \mu, \varphi) = R(\mu) I(x, y, 0, -\mu, \varphi) + \delta(\mu - \mu_0)\delta(\varphi - \varphi_0)\delta(x)\delta(y) \tag{2}$$

where $R(\mu)$ is the Fresnel reflection coefficient.

For Lambert's law of reflection, we have

$$I(x, y, 0, \mu, \varphi) = \frac{R_d(\mu)}{\pi}\int_0^{2\pi} d\varphi' \int_0^1 d\mu' \mu' I(x, y, 0, -\mu', \varphi') + \delta(\mu - \mu_0)\delta(\varphi - \varphi_0)\delta(x)\delta(y) \tag{3}$$

and in both cases $0 < \mu < 1$, $0 < \varphi < 2\pi$. In equation (3) $R_d(\mu)$ is the fraction of radiation reflected internally. In what follows $R_d(\mu)$ will be assumed to be independent of μ. The delta functions in (2) and (3) denote the mono-directional nature of the beam and the fact that it strikes the surface at $x = 0$ and $y = 0$. It is of passing interest to note that Elliott's (1952, 1955) point source representation is equivalent to an incident source of the form

$$I(x, y, 0, \mu, \varphi) = \frac{S_0}{2\mu}\delta(x)\delta(y).$$

If we define the infinite medium Fourier transform in the x and y directions, we find

$$\left(\mu\frac{\partial}{\partial z} + i\sqrt{1-\mu^2}\{k_1\cos\varphi + k_2\sin\varphi\} + 1\right)\bar{I}(k_1, k_2, z, \mu, \varphi) = \frac{c}{4\pi}\bar{I}_0(k_1, k_2, z) \tag{4}$$

where

$$\bar{I}(k_1, k_2, z, \mu, \varphi) = \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}k_1 x} \int_{-\infty}^{\infty} \mathrm{d}y\, \mathrm{e}^{-\mathrm{i}k_2 y} I(x, y, z, \mu, \varphi). \tag{5}$$

The associated boundary conditions (2) and (3) become

$$\bar{I}(k_1, k_2, 0, \mu, \varphi) = R(\mu)\bar{I}(k_1, k_2, 0, -\mu, \varphi) + \delta(\mu - \mu_0)\delta(\varphi - \varphi_0) \tag{6}$$

$$\bar{I}(k_1, k_2, 0, \mu, \varphi) = \frac{R_d(\mu)}{\pi} \int_0^{2\pi} \mathrm{d}\varphi' \int_0^1 \mathrm{d}\mu' \mu' \bar{I}(k_1, k_2, 0, -\mu', \varphi') + \delta(\mu - \mu_0)\delta(\varphi - \varphi_0). \tag{7}$$

Now in Williams (1982) we showed, by using the Wiener–Hopf technique (Williams 1971) that the solution of equation (4) at the surface $z = 0$, subject to the boundary condition where $R(\mu) = R_d = 0$, i.e. no internal reflection, is given by

$$\bar{I}(k_1, k_2, 0, -\mu, \varphi) = \frac{c}{4\pi} \frac{\mu_0 H\left(\frac{\mu_0}{1+\mathrm{i}f_0}\right) H\left(\frac{\mu}{1+\mathrm{i}f}\right)}{\mu_0(1 + \mathrm{i}f) + \mu(1 + \mathrm{i}f_0)} \tag{8}$$

where $f = \sqrt{1 - \mu^2}(k_1 \cos\varphi + k_2 \sin\varphi)$ and f_0 is the same but with $\mu = \mu_0,\ \varphi = \varphi_0$. The function $H(1/p)$ in equation (8), is a generalization of the Chandrasekhar H-function, details of which are given in appendix A.

If we now write the new boundary conditions (6) and (7) with reflection in the form

$$\bar{I}(k_1, k_2, 0, \mu, \varphi) = \Phi(k_1, k_2, \mu, \varphi) + \delta(\mu - \mu_0)\delta(\varphi - \varphi_0) \tag{9}$$

then by the principle of superposition (Williams 2006), we may write the general solution for this new boundary condition as

$$\begin{aligned}\bar{I}(k_1, k_2, 0, -\mu, \varphi) = \frac{c}{4\pi} H\left(\frac{\mu}{1 + if}\right) &\left\{ \frac{\mu_0 H\left(\frac{\mu_0}{1+\mathrm{i}f_0}\right)}{\mu_0(1 + \mathrm{i}f) + \mu(1 + \mathrm{i}f_0)} \right.\\ &\left. + \int_0^{2\pi} \mathrm{d}\varphi' \int_0^1 \mathrm{d}\mu' \mu' \Phi(k_1, k_2, \mu', \varphi') \frac{H\left(\frac{\mu'}{1+\mathrm{i}f'}\right)}{\mu'(1 + \mathrm{i}f) + \mu(1 + \mathrm{i}f')} \right\}.\end{aligned} \tag{10}$$

The arguments behind this principle of superposition are given in appendix C.

For specular reflection,

$$\Phi(k_1, k_2, \mu, \varphi) = R(\mu)\bar{I}(k_1, k_2, 0, -\mu, \varphi) \tag{11a}$$

and for Lambert's law

$$\Phi(k_1, k_2, \mu, \varphi) = \frac{R_d}{\pi} \int_0^{2\pi} \mathrm{d}\varphi' \int_{-1}^1 \mathrm{d}\mu' \mu' \bar{I}\big(k_1, k_2, 0, -\mu', \varphi'\big). \tag{11b}$$

In the case of (11b), equation (10) may be solved explicitly for $\bar{I}(\ldots)$ as we will show below. However, for specular reflection, (11a), we must solve a Fredholm integral equation. There are strong similarities between equations (10) and (14) in Crosbie and Dougherty (1988), although our equation also applies to the Lambert law. In addition, Crosbie and Dougherty do not have an explicit analytical expression for the generalized H-function $H(\mu/(1 + \mathrm{i}f))$ and also consider only the case of normal incidence for which $f = 0$ and is therefore less general.

We also showed in Williams (1982) that the surface scalar intensity $\bar{I}_0(k_1, k_2, 0)$ may be written as

$$\bar{I}_0(k_1, k_2, 0) = H\left(\frac{\mu_0}{1 + \mathrm{i}f_0}\right). \tag{12}$$

From superposition this becomes

$$\bar{I}_0(k_1, k_2, 0) = H\left(\frac{\mu_0}{1 + \mathrm{i}f_0}\right) + \int_0^{2\pi} \mathrm{d}\varphi \int_0^1 \mathrm{d}\mu \Phi(k_1, k_2, \mu, \varphi) H\left(\frac{\mu}{1 + \mathrm{i}f}\right). \tag{13}$$

The inverse Fourier transform of (10) can only be obtained explicitly in certain circumstances and we shall discuss this matter below.

3. Special cases of the solution

3.1. Specular reflection

The specular case is the most difficult one as it leads to an integral equation. We shall consider that case first and obtain some parameters of interest. Let us note, however, that for Fresnel reflection, refraction takes place for radiation as the wave passes through the surface and Snell's law describes this effect (Born and Wolf 1999). We have defined the incident beam as mono-directional, but it could be of the general form $\psi(\mu, \varphi)$. In this case, and bearing in mind the refraction, the actual source entering the medium will be (Williams 2006)

$$q(\mu, \varphi) = n^2(1 - R(\mu))\psi(\sqrt{1 - n^2(1 - \mu^2)}, \varphi) \tag{14}$$

where n is the refractive index of the medium. If $\psi(\mu, \varphi)$ is mono-directional as defined in the boundary conditions, such that $\psi(\mu, \varphi) = \delta(\mu - \mu^*)\delta(\varphi - \varphi^*)$then

$$q(\mu, \varphi) = (1 - R(\bar{\mu}^*))\frac{\mu^*}{\bar{\mu}^*}\delta(\mu - \bar{\mu}^*)\delta(\varphi - \varphi^*) \tag{15}$$

with $\bar{\mu}^* = (1 - (1 - \mu^{*2})/n^2)^{1/2}$. Thus the actual solution is given by weighting equation (10) with $q(\mu, \varphi)$ in the form

$$\int_0^{2\pi} \mathrm{d}\varphi_0 \int_{-1}^1 \mathrm{d}\mu_0 q(\mu_0, \varphi_0)\bar{I}(k_1, k_2, 0, -\mu, \varphi) \tag{16}$$

where the variables (μ_0, φ_0) are implicit in $\bar{I}(\ldots)$. We will use this representation later. Crosbie and Dougherty (1988) have also employed this type of source term but only for normal incidence in which $\mu^* = 1$.

Before dealing with the general case, we note that for $R = 0$, we can reduce the expressions for the surface flux and current to convenient forms for numerical evaluation. Now in Williams (1982), we showed that certain integrals over (μ, φ) could be reduced, under particular circumstances, to a simpler form, i.e.

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\varphi \int_{-1}^1 \mathrm{d}\mu G\left(\frac{\mu}{1 + if}\right) = \int_0^1 \mathrm{d}w G\left(\frac{w}{\sqrt{1 + k^2 w^2}}\right) \tag{17}$$

where $k^2 = k_1^2 + k_2^2$. This is a useful reduction and applying it to equation (12) we find for an isotropic beam (i.e. averaged over all possible μ_0 and φ_0)

$$\bar{I}_0(k, 0) = 2\pi \int_0^1 \mathrm{d}w H\left(\frac{w}{\sqrt{1 + k^2 w^2}}\right) \tag{18}$$

and from equation (8), the net outward current is

$$\bar{J}(k, 0) = \pi c \int_0^1 \mathrm{d}w \bar{w} H(\bar{w}) \int_0^1 \mathrm{d}w_0 \frac{\bar{w}_0 H(\bar{w}_0)}{\bar{w} + \bar{w}_0} \tag{19}$$

where $\bar{w} = w/\sqrt{1 + k^2 w^2}$ and $\bar{w}_0 = w_0/\sqrt{1 + k^2 w_0^2}$. Unfortunately, the pseudo-H function $H(\bar{w})$ does not satisfy any convenient equations which enable (19) to be simplified as in the case for $k = 0$.

We still need to invert the Fourier transform, but because of symmetry we note that the surface intensity and current depend only on the radial direction $\rho = \sqrt{x^2 + y^2}$. Thus we find

$$I_0(\rho, 0) = \int_0^\infty \mathrm{d}k k J_0(k\rho) \int_0^1 \mathrm{d}w H(\bar{w}) \tag{20}$$

with a similar expression for the current $J(\rho, 0)$. In equation (20) $J_0(x)$ is a Bessel function. The albedo is defined as $\alpha(\rho) = J(\rho, 0)/\pi$. These expressions will be evaluated numerically in a later section. It is useful, however, to evaluate the asymptotic forms of $I_0(\rho, 0)$ and $J(\rho, 0)$. We may do this by expanding $H(\bar{w})$ in terms of k. Only the case $c = 1$, i.e. no absorption, will be considered.

As we can see from appendix A, $H(\bar{w}) = H_0(w)(1 - kw) + O(k^2)$, where H_0 is the conventional Chandrasekhar H-function in which $k = 0$. Using this in equation (19) and expanding to O(k), we find

$$\bar{J}(k, 0) = \tfrac{1}{2}\left(1 - kh_1^2\right) + O(k^2) \tag{21}$$

where

$$h_n = \int_0^1 \mathrm{d}w w^n H_0(w). \tag{22}$$

Also

$$\bar{I}_0(k, 0) = h_0 - kh_1 + O(k^2). \tag{23}$$

Using the inversion formula and $h_0 = 2$, $h_1 = 2/\sqrt{3}$, we find

$$J(\rho, 0) \sim \frac{2}{3\rho^3} \tag{24}$$

and

$$I_0(\rho, 0) \sim \frac{2}{\sqrt{3}\rho^3} \tag{25}$$

from which we have the classic relation $I_0(\rho, 0) \sim \sqrt{3} J(\rho, 0)$ (Elliott 1952).

Of course for the case of specific values of μ_0 and φ_0, the above results are not so simple since then the intensity and current will depend on the azimuthal angular coordinate Θ about the z-axis as well as ρ, i.e., we must seek $I_0(\rho, \Theta, 0)$. This is difficult to evaluate numerically as it contains H-functions with complex arguments.

Another quantity of physical interest is the mean distance of travel in the x-direction, i.e.

$$\langle x(\mu_0, \varphi_0)\rangle = \frac{\int_{-\infty}^{\infty} \mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}x x I_0(x, y, 0)}{\int_{-\infty}^{\infty} \mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}x I_0(x, y, 0)}. \tag{26}$$

This can be evaluated directly in terms of the Fourier transform, namely:

$$\langle x(\mu_0, \varphi_0)\rangle = \frac{\mathrm{i}}{\bar{I}_0(0, 0, 0)} \frac{\partial}{\partial k_1} \bar{I}_0(k_1, 0, 0)\bigg|_{k_1=0} \tag{27}$$

where $\bar{I}_0$ is given by equation (13). We shall now demonstrate how to calculate this quantity for specular reflection. First we must consider equation (10) with $\Phi = R\bar{I}$, i.e. equation (11*a*). Then we expand $\bar{I}$ in powers of k_1 and k_2 as follows:

$$\bar{I}(k_1, k_2, 0, -\mu, \varphi) = \frac{1}{2\pi}[\hat{I}_0(\mu, \varphi) + \mathrm{i}k_1 \hat{I}_1(\mu, \varphi) + \mathrm{i}k_2 \hat{I}_2(\mu, \varphi) + \cdots]. \tag{28}$$

Also we note from appendix A that

$$\frac{\mu_0 H(\bar{\mu}_0) H(\bar{\mu})}{\mu_0(1 + \mathrm{i}f) + \mu(1 + \mathrm{i}f_0)} = \frac{\mu_0 H_0(\mu_0) H_0(\mu)}{\mu + \mu_0} [1 - \mathrm{i}Z(\mu, \mu_0)(k_1 \cos\varphi_0 + k_2 \sin\varphi_0) - \mathrm{i}Z(\mu_0, \mu)(k_1 \cos\varphi + k_2 \sin\varphi) + \cdots] \tag{29}$$

where Z is defined in appendix A and $\bar{\mu} = \mu/(1 + if)$. Inserting these expansions into equation (10) and collecting up coefficients of 1, k_1, k_2, we find

$$2\hat{I}_0(\mu) = G(\mu, \mu_0) + \int_0^1 \mathrm{d}\mu' R(\mu')G(\mu, \mu')\hat{I}_0(\mu') \tag{30}$$

where $\hat{I}_0(\mu) = \hat{I}_0(\mu, \varphi)$ is independent of φ.

$$2\hat{I}_1(\mu, \varphi) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\varphi' \int_0^1 \mathrm{d}\mu' R(\mu')G(\mu, \mu')\hat{I}_1(\mu', \varphi') - G(\mu, \mu_0)\,[Z(\mu, \mu_0)\cos\varphi_0 + Z(\mu_0, \mu)\sin\varphi_0] - \cos\varphi \int_0^1 \mathrm{d}\mu' R(\mu')G(\mu, \mu')Z(\mu', \mu)\hat{I}_0(\mu') \tag{31}$$

$$2\hat{I}_2(\mu, \varphi) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\varphi' \int_0^1 \mathrm{d}\mu' R(\mu')G(\mu, \mu')\hat{I}_2(\mu', \varphi') - G(\mu, \mu_0)\,[Z(\mu, \mu_0)\sin\varphi_0 + Z(\mu_0, \mu)\cos\varphi_0] - \sin\varphi \int_0^1 \mathrm{d}\mu' R(\mu')G(\mu, \mu')Z(\mu', \mu)\hat{I}_0(\mu') \tag{32}$$

with

$$G(\mu, \mu') = \frac{\mu' H_0(\mu)H_0(\mu')}{\mu + \mu'} \tag{33}$$

These are three coupled integral equations for $\hat{I}_0$, $\hat{I}_1$, $\hat{I}_2$. To get $\bar{I}_0(k_1, k_2, 0)$ in equation (13), we have

$$\bar{I}_0(k_1, k_2, 0) = \hat{I}_{00} + \mathrm{i}k_1\hat{I}_{10} + \mathrm{i}k_2\hat{I}_{20} + \cdots. \tag{34}$$

From which, using equation (27),

$$\langle x(\mu_0, \varphi_0)\rangle = -\frac{\hat{I}_{10}}{\hat{I}_{00}}. \tag{35}$$

Hence from (13)

$$\hat{I}_{00} = H_0(\mu_0) + \int_0^1 \mathrm{d}\mu R(\mu)H_0(\mu)\hat{I}_0(\mu) \tag{36}$$

and

$$-\hat{I}_{10} = H_0(\mu_0)\sqrt{1-\mu_0^2}\left[\frac{\mu_0}{1+\mu_0} + \mu_0\hat{\Omega}(\mu_0)\right]\cos\varphi_0 - \int_0^1 \mathrm{d}\mu R(\mu)H_0(\mu)I_1^{(0)}(\mu) \tag{37}$$

where

$$\hat{I}_1(\mu, \varphi) = I_1^{(0)}(\mu) + I_1^{(1)}(\mu)\cos\varphi \tag{38}$$

and

$$2I_1^{(0)}(\mu) = -G(\mu, \mu_0)Z(\mu, \mu_0)\cos\varphi_0 + \int_0^1 \mathrm{d}\mu' R(\mu')G(\mu, \mu')I_1^{(0)}(\mu') \tag{39}$$

$\hat{\Omega}$ is defined in appendix A. So to obtain $\langle x\rangle$ we need to evaluate the integral equation (30) for $\hat{I}_0(\mu)$ and hence from (36) get $\hat{I}_{00}$. Then we solve equation (39) and hence from (37) get $\hat{I}_{10}$. Using an analogous technique leads to equations for $\langle y\rangle$. If we now apply the source operator (16) to $\hat{I}_{00}$ and $\hat{I}_{10}$, we find for $\varphi_0 = 0$

$$\hat{I}_{00} = (1 - \tilde{R}(\mu^*))\frac{\mu^*}{\bar{\mu}^*}H_0(\bar{\mu}^*) + \int_0^1 \mathrm{d}\mu R(\mu)H_0(\mu)\hat{I}_0(\mu) \tag{40}$$

$$-\hat{I}_{10} = (1 - \tilde{R}(\mu^*))\frac{\mu^*}{\bar{\mu}^*} H_0(\bar{\mu}^*)\sqrt{1-\bar{\mu}^{*2}}\left[\frac{\bar{\mu}^*}{1+\bar{\mu}^*} + \bar{\mu}^*\hat{\Omega}(\bar{\mu}^*)\right]$$
$$- \int_0^1 d\mu R(\mu) H_0(\mu) I_1^{(0)}(\mu). \tag{41}$$

The associated integral equations become

$$2\hat{I}_0(\mu) = (1 - \tilde{R}(\mu^*))\frac{\mu^*}{\bar{\mu}^*} G(\mu, \bar{\mu}^*) + \int_0^1 d\mu' R(\mu') G(\mu, \mu')\hat{I}_0(\mu') \tag{42}$$

and

$$2I_1^{(0)}(\mu) = -(1 - \tilde{R}(\mu^*))\frac{\mu^*}{\bar{\mu}^*} G(\mu, \bar{\mu}^*) Z(\mu, \bar{\mu}^*) + \int_0^1 d\mu' R(\mu') G(\mu, \mu') I_1^{(0)}(\mu'). \tag{43}$$

We have set $\varphi_0 = 0$, because it fixes the plane in which the beam lies to be co-incident with the x-axis. In that case $\langle y \rangle = 0$ because of symmetry in the y-direction. $\tilde{R}(\mu^*)$ is the complementary Fresnel coefficient (Born and Wolf 1999). Numerical work will be presented below.

3.2. Lambert reflection

In this case we choose equation (11*b*) for Φ, whence the equation for $\bar{I}(k_1, k_2, 0, -\mu, \varphi)$ becomes

$$\bar{I}(k_1, k_2, 0, -\mu, \varphi) = \frac{c}{4\pi} H(\bar{\mu}) \left\{ \frac{\mu_0 H(\bar{\mu}_0)}{\mu_0(1 + \mathrm{i}f) + \mu(1 + \mathrm{i}f_0)} \right.$$
$$\left. + R_d \bar{J}_0 \int_0^{2\pi} d\varphi' \int_0^1 d\mu' \mu' \frac{H(\bar{\mu}')}{\mu'(1 + \mathrm{i}f) + \mu(1 + \mathrm{i}f')} \right\} \tag{44}$$

where $\bar{\mu} = \mu/(1 + \mathrm{i}f)$ and

$$\bar{J}_0 = \frac{1}{\pi} \int_0^{2\pi} d\varphi \int_0^1 d\mu \mu \bar{I}(k_1, k_2, 0, -\mu, \varphi). \tag{45}$$

If now we integrate (44) over all $\mu(0, 1)$ and $\varphi(0, 2\pi)$ and use the special transformation (17), we find after integrating over μ_0 and φ_0 (assuming isotropic beam source), that

$$\bar{J}_0 = \frac{c\bar{F}(k)}{1 - cR_d\bar{F}(k)} \tag{46}$$

where

$$\bar{F}(k) = \int_0^1 dw \bar{w} H(\bar{w}) \int_0^1 dw_0 \frac{\bar{w}_0 H(\bar{w}_0)}{\bar{w} + \bar{w}_0}. \tag{47}$$

In view of the fact that only a fraction $1 - R_d$ is transmitted through the surface from the source and a fraction $1 - R_d$ is re-transmitted from the medium, the Fourier transform of the albedo can be written as

$$\bar{\alpha}(k) = (1 - R_d)^2 \bar{J}_0 + R_d \tag{48}$$

where the last term is the fraction directly reflected from the surface not having entered the medium. The inverse transform is therefore

$$\alpha(\rho) = (1 - R_d)^2 \frac{1}{2\pi} \int_0^\infty dk k J_0(k\rho) \frac{c\bar{F}(k)}{1 - cR_d\bar{F}(k)} + R_d \frac{\delta(\rho)}{2\pi\rho}. \tag{49}$$

The surface intensity for an isotropic incident beam is from equation (13)

$$\bar{I}_0(k, 0) = \frac{2\pi}{1 - cR_d\bar{F}(k)} \int_0^1 dw H(\bar{w}) \tag{50}$$

whence

$$I_0(\rho, 0) = \int_0^\infty \frac{\mathrm{d}k k J_0(k\rho)}{1 - cR_d \bar{F}(k)} \int_0^1 \mathrm{d}w H(\bar{w}). \tag{51}$$

These expressions will be evaluated numerically below, but we do note that for large ρ (small k) we can write for $c = 1$,

$$\alpha(\rho) \sim \frac{2}{3\pi\rho^3} \tag{52}$$

and

$$I_0(\rho, 0) \sim \frac{2}{\sqrt{3}(1 - R_d)\rho^3}\left(1 + \frac{4R_d}{\sqrt{3}(1 - R_d)}\right). \tag{53}$$

Another case which is relatively easy to handle is when the beam is normal to the surface, i.e. $\mu_0 = 1$. Then we have from (44) the Fourier transform of the albedo as

$$\bar{\alpha}(k) = \pi \bar{J}_0 = (1 - R_d)^2 \frac{c}{2} \frac{H(1)\bar{G}(k)}{1 - cR_d \bar{F}(k)} + R_d \tag{54}$$

where

$$\bar{G}(k) = \int_0^1 \mathrm{d}w \frac{\bar{w}H(\bar{w})}{1 + \bar{w}} \tag{55}$$

and from (13)

$$\bar{I}_0(k, 0) = H(1)\left[1 + \frac{cR_d \bar{G}(k)\bar{H}(k)}{1 - cR_d \bar{F}(k)}\right] \tag{56}$$

with

$$\bar{H}(k) = \int_0^1 \mathrm{d}w H(\bar{w}). \tag{57}$$

Using the same technique as for equations (52) and (53) we find

$$\alpha(\rho) \sim \frac{1}{2\pi\rho^3}\left[\frac{4}{3}R_d + \frac{1}{\sqrt{3}}(1 - R_d)H_0(1)\right] \tag{58}$$

and

$$I_0(\rho, 0) \sim \frac{1}{2\pi\rho^3}\left[\frac{4R_d}{\sqrt{3}(1 - R_d)} + H_0(1)\left\{1 + \frac{4R_d}{\sqrt{3}(1 - R_d)} + \frac{16R_d^2}{(1 - R_d)^2}\right\}\right]. \tag{59}$$

We stress that these results apply only for $c = 1$. We also note that the albedo, $\alpha(\rho)$, is not the albedo in the normal sense, which we expect to be less than unity; we might call it a localized albedo. However, for $c = 1$, it does satisfy the conservation condition

$$2\pi \int_0^\infty \mathrm{d}\rho\rho\alpha(\rho) = 1$$

which may be verified by setting $k = 0$ in $\bar{\alpha}(k)$.

4. Revisiting Williams (1982)

In our earlier work of 1982 (henceforth called W1), we also included a section on the case of a line source in a half-space along the z-axis. We now wish to take the opportunity of updating the accuracy of some of the tables and to correct some errors. The result to be evaluated was the surface intensity in the form

$$W(\rho, 0) = \frac{E_0(1 - \alpha)}{4\pi c} \int_0^\infty \mathrm{d}k k J_0(k\rho)\{[1 - (c/k)\tan^{-1} k]^{-1/2} - 1\} \tag{60}$$

and also the approximate expression based upon the H_1 of Rybicki (1971) which has the form

$$W(\rho, 0) = \frac{E_0(1-\alpha)}{4\pi c} \int_0^\infty \mathrm{d}k\, k\, J_0(k\rho) \left\{ \left(\frac{3+k^2}{3(1-c)+k^2} \right)^{1/2} - 1 \right\}. \tag{61}$$

In order to evaluate equation (60), we found it expedient to consider the following Fourier transform

$$F(y) = \int_{-\infty}^{\infty} \mathrm{d}k\, \mathrm{e}^{\mathrm{i}ky} \{[1 - (c/k)\tan^{-1} k]^{-1/2} - 1\}. \tag{62}$$

In view of the fact that $1 - (c/k)\tan^{-1} k = 0$ has roots at $k = \pm \mathrm{i}\nu$, where $0 \leqslant \nu < 1$, there are branch points at $k = \pm \mathrm{i}\nu$ due to the square root. In addition, there are the usual branch points at $k = \pm \mathrm{i}$ due to the singularities of the *arctan* function. Thus the contour must be deformed around a cut from $\mathrm{i}\nu$ to $\mathrm{i}\infty$ for $y > 0$. Carrying out the algebra, which is tedious but straightforward, we find

$$\begin{aligned} F(y) = 2 \int_\nu^1 & \frac{\mathrm{d}t\, \mathrm{e}^{-t|y|}}{\left[\frac{c}{2t} \log\left(\frac{1+t}{1-t}\right) - 1\right]^{1/2}} \\ & + \sqrt{2} \int_1^\infty \mathrm{d}t\, \mathrm{e}^{-t|y|} \left[g(c,t)^{1/2} - \left(1 - \frac{c}{2t} \log\left(\frac{t+1}{t-1}\right)\right) g(c,t) \right]^{1/2} \end{aligned} \tag{63}$$

where

$$\frac{1}{g(c,t)} = \left(1 - \frac{c}{2t} \log\left(\frac{t+1}{t-1}\right)\right)^2 + \left(\frac{c\pi}{2t}\right)^2. \tag{64}$$

More concisely, we have

$$F(y) = \int_\nu^\infty \mathrm{d}t\, G(t)\, \mathrm{e}^{-t|y|} \tag{65}$$

where $G(t)$ is defined by the integrand of equation (63). In our earlier work, we omitted the contribution to $G(t)$ from $(1, \infty)$.

From (62) we may write the inverse transform as

$$[1 - (c/k)\tan^{-1} k]^{-1/2} - 1 = \frac{1}{2\pi} \int_{-\infty}^\infty \mathrm{d}y\, \mathrm{e}^{-\mathrm{i}ky} F(y) = \frac{1}{\pi} \int_\nu^\infty \frac{\mathrm{d}t\, t\, G(t)}{t^2 + k^2}. \tag{66}$$

In fact the quantity of interest is

$$\int_0^\infty \mathrm{d}k\, k\, J_0(k\rho) \{[1 - (c/k)\tan^{-1} k]^{-1/2} - 1\} = \frac{1}{\pi} \int_\nu^\infty \mathrm{d}t\, t\, G(t) \int_0^\infty \frac{\mathrm{d}k\, k\, J_0(k\rho)}{t^2 + k^2}. \tag{67}$$

But the integral over k can be simplified and we find that (67) reduces to

$$\frac{1}{\pi} \int_\nu^\infty \mathrm{d}t\, t\, G(t) K_0(t\rho) \tag{68}$$

where $K_0(x)$ is the modified Bessel function. This particular form, containing the $K_0(x)$ function, converges far more rapidly than the left-hand side of equation (67). In order to correct table 1 in W1, we present table 1 which gives expression (68). The equivalent expression for the H_1 approximation is correct in W1, but the numerical values have been amended in table 1 and are headed 'approx'.

In W1, we also calculated $\langle x \rangle$, the mean distance of travel of a photon or neutron in the x-direction. Equation (123) of W1 is correct but the values in table 2 of W1 are not very accurate. Table 2 gives values correct to 4 significant figures. We also give figure 1 which illustrates the variation of $\langle x \rangle$ with the cosine of the incident angle μ_0 and with $\varphi_0 = 0$. The variation of $\langle x(\mu_0, 0) \rangle$ is of interest since it increases from zero at $\mu_0 = 0$, i.e. grazing incidence, goes through a maximum, and then reduces to zero at $\mu_0 = 1$, i.e. normal incidence.

Table 1. Surface intensity due to line source in a semi-infinite medium. Exact values and those from an approximate H-function.

	$c = 1$		$c = 0.9$		$c = 0.5$	
ρ	Exact	Approx	Exact	Approx	Exact	Approx
0.1	9.461	4.600	7.590	3.449	3.548	1.596
0.2	5.239	3.580	3.858	2.538	1.611	1.099
0.3	3.758	2.999	2.570	2.022	0.9763	0.8251
0.4	2.982	2.599	1.907	1.671	0.6662	0.6438
0.5	2.497	2.300	1.499	1.411	0.4850	0.5138
0.6	2.160	2.063	1.221	1.209	0.3679	0.4162
0.7	1.910	1.871	1.019	1.046	0.2871	0.3408
0.8	1.717	1.711	0.8660	0.9131	0.2288	0.2813
0.9	1.562	1.575	0.7457	0.8021	0.1852	0.2336
1.0	1.434	1.458	0.6488	0.7084	0.1518	0.1950
1.5	1.024	1.056	0.3574	0.4028	0.063 43	0.083 26
2.0	0.7980	0.8215	0.2165	0.2437	0.029 82	0.037 49
2.5	0.6534	0.6688	0.1379	0.1534	0.014 95	0.017 42
3.0	0.5524	0.5625	0.090 77	0.099 26	0.007 801	0.008 265
3.5	0.4779	0.4846	0.061 06	0.065 59	0.004 185	0.003 983
4.0	0.4207	0.4252	0.041 75	0.044 05	0.002 291	0.001 943
4.5	0.3754	0.3786	0.028 90	0.030 00	0.001 274	0.000 957
5.0	0.3387	0.3410	0.020 21	0.020 62	0.000 7167	0.000 4750

Table 2. Mean distance of travel in the x-direction as a function of incident beam angle for the searchlight problem with no internal reflection.

μ_0	$c = 1$	$c = 0.9$	$c = 0.5$
0.01	0.028 68	0.021 75	0.010 15
0.05	0.1027	0.072 39	0.031 05
0.1	0.1696	0.1133	0.045 81
0.2	0.2650	0.1638	0.061 06
0.3	0.3292	0.1910	0.067 07
0.4	0.3705	0.2032	0.068 05
0.5	0.3922	0.2044	0.065 76
0.6	0.3949	0.1962	0.061 02
0.7	0.3774	0.1794	0.054 14
0.8	0.3352	0.1528	0.044 90
0.9	0.2551	0.1117	0.032 06
0.95	0.1865	0.080 08	0.022 74
0.97	0.1463	0.062 35	0.017 63
0.99	0.085 53	0.036 18	0.010 19
1.0	0.0	0.0	0.0

This behaviour is expected because at normal incidence $\langle x \rangle = 0$ from symmetry ($\langle x^2 \rangle$ is of course nonzero), and at grazing incidence most of the particles (photons) are near the surface and leak out preferentially. All the results here are for $R = R_d = 0$, i.e. no internal reflection.

5. Numerical results and discussion for the searchlight problem

In this section, we will illustrate some of the general results given above numerically. Firstly, we consider $\langle x(\mu_0, \varphi_0) \rangle$ as defined by equation (35) and the associated integral equations (42) and (43). These integral equations are solved numerically by means of the NAG library routine

Table 3. Mean distance of travel in the x-direction, $\bar{x}(\mu_0, 0)$, for different values of the refractive index n, $c = 1$.

μ_0	$n = 1$	$n = 1.1$	$n = 4/3$
0.0	0.0	0.4680	0.5454
0.1	0.1696	0.4695	0.5434
0.2	0.2650	0.4729	0.5370
0.3	0.3292	0.4751	0.5259
0.4	0.3705	0.4728	0.5090
0.5	0.3922	0.4628	0.4851
0.6	0.3949	0.4421	0.4524
0.7	0.3774	0.4070	0.4077
0.8	0.3352	0.3517	0.3459
0.9	0.2551	0.2621	0.2538
1.0	0.0	0.0	0.0

D05ABF. In this routine, the solution $f(x)$is expanded as an n-term Chebyshev series in the form

$$f(x) = \frac{1}{2}c_1 T_1(x) + \sum_{i=2}^{n} c_i T_i(x). \tag{69}$$

The coefficients c_i, for $i = 1, 2, \ldots, n$, of this series are determined directly from approximate values f_i, for $i = 1, 2, \ldots, n$, of the function $f(x)$ at the first n of a set of $m + 1$ Chebyshev points

$$x_i = \tfrac{1}{2}[a + b + (b - a)\cos([i - 1]\pi/m)], \qquad i = 1, 2, \ldots, m + 1 \tag{70}$$

where $b = 1$ and $a = 0$ are the limits on the integral. The values f_i are obtained by solving a set of simultaneous linear algebraic equations formed by applying a quadrature formula to the integral at each of the above points (Clenshaw and Curtis 1960).

Figure 2 shows the variation of $\langle x(\mu_0, \varphi_0)\rangle$ for four cases. If the refractive index $n = 1$, we have the result given in W1 which is the same as the $c = 1$ case in figure 1 but evaluated by using an entirely different method. More important is the case when $n > 1$ and we have specular-Fresnel reflection; the results shown are very different. For example, for $n = 1$, $\langle x(\mu_0, \varphi_0)\rangle$ has a maximum and is zero at $\mu_0 = 0,\ 1$. For $n > 1$, however, we find that $\langle x(\mu_0, \varphi_0)\rangle$ is finite at $\mu_0 = 0$, i.e. grazing incidence, and zero at normal incidence. The reason for this behaviour is that at grazing incidence, due to refraction, the incident beam enters the medium at an angle $\vartheta_c = \sin^{-1}(1/n)$ and can therefore travel some distance in the x-direction before photons leak out. For $n > 4/3$, the maximum appears to be at $\mu_0 = 0$. Table 3 gives some representative values of $\langle x(\mu_0, \varphi_0)\rangle$ for $n = 1$, 1.1 and 4/3. Figure 2 shows that the maximum in $\langle x(\mu_0, \varphi_0)\rangle$ moves to lower values of μ_0 as n increases, but for $n > 1$ the value at $\mu_0 = 0$ is always greater than zero, as we expect on the basis of the argument given above concerning refraction.

We now consider the values of the albedo $\alpha(\rho)$ and surface intensity $I_0(\rho, 0)$ for the point isotropic and normal beam sources. This will be done for the case of Lambert internal reflection as described in section 3.2. All values are for $c = 1$, although $c < 1$ poses no problem numerically. For convenience we plot $A_0\rho^3\alpha(\rho)$ and $A_1\rho^3 I_0(\rho, 0)$ in figures 3 and 4, respectively. Note that the integrals over the Bessel function $J_0(x)$ are evaluated by dividing the integral into sections covering the distance between the roots of $J_0(x) = 0$. The resulting terms, which alternate in sign, are summed using the Shanks method (Bender and Orszag 1978) which gives excellent convergence, with not more than ten terms needed unless ρ is

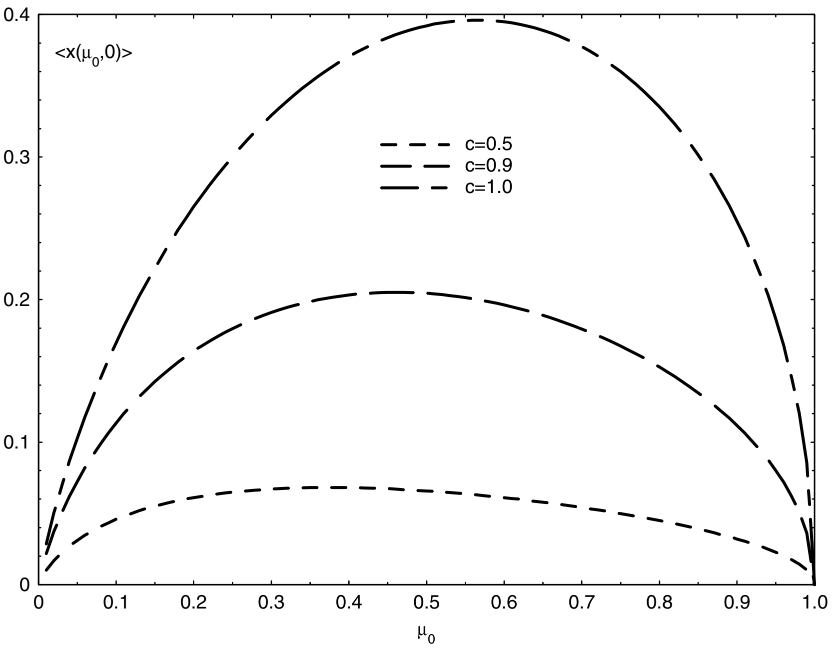

Figure 1. Mean distance of travel in the *x*-direction.

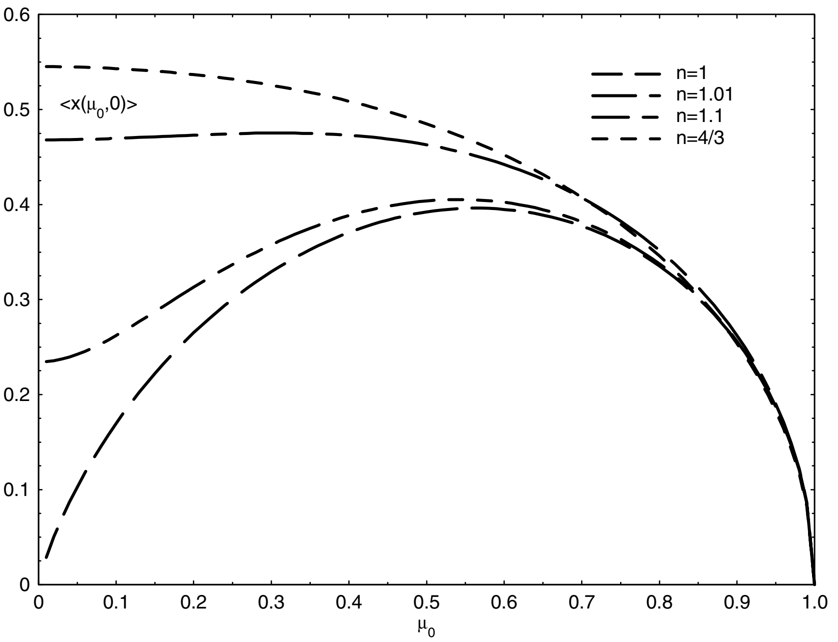

Figure 2. Mean value of x as a function of the cosine of the incident angle, $c = 1$.

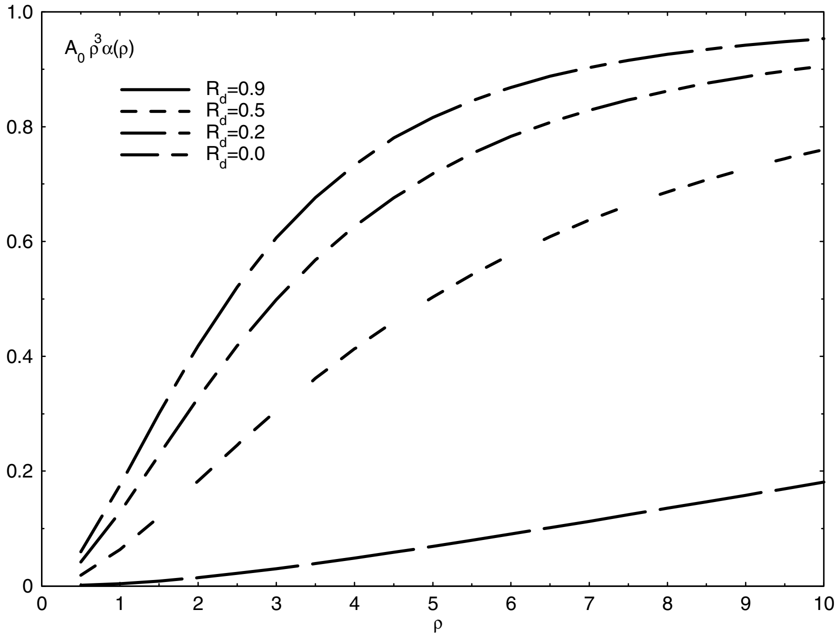

Figure 3. Normalized albedo for isotropic beam.

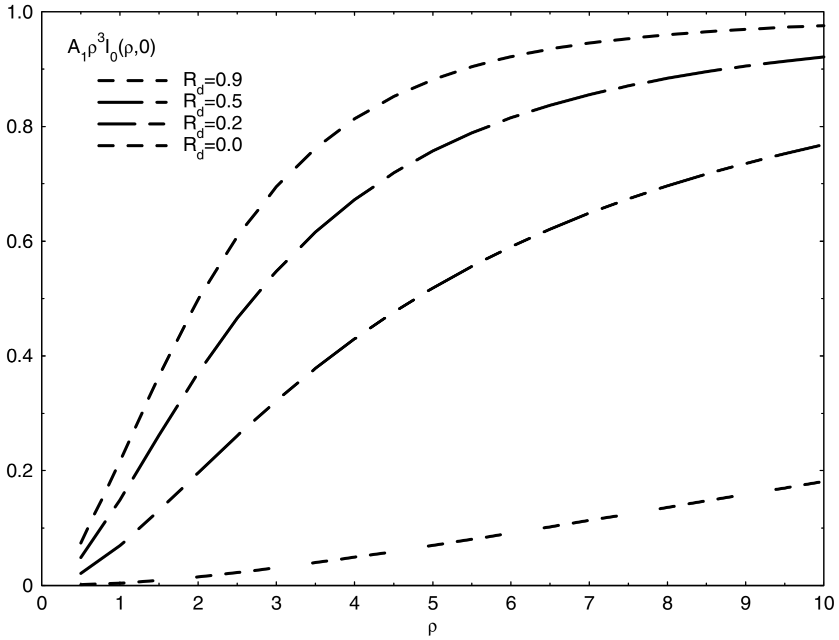

Figure 4. Normalized surface intensity for isotropic beam.

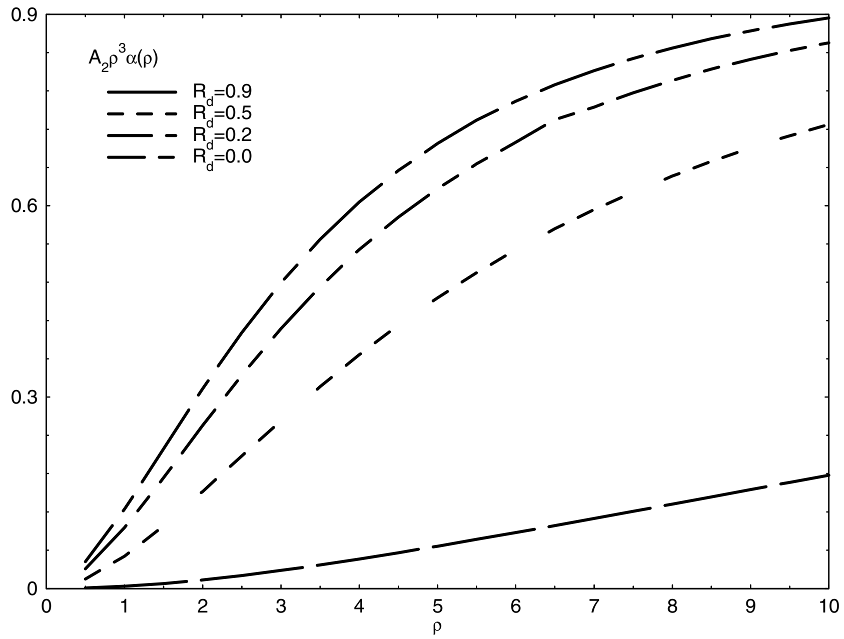

Figure 5. Normalized albedo for a normal beam.

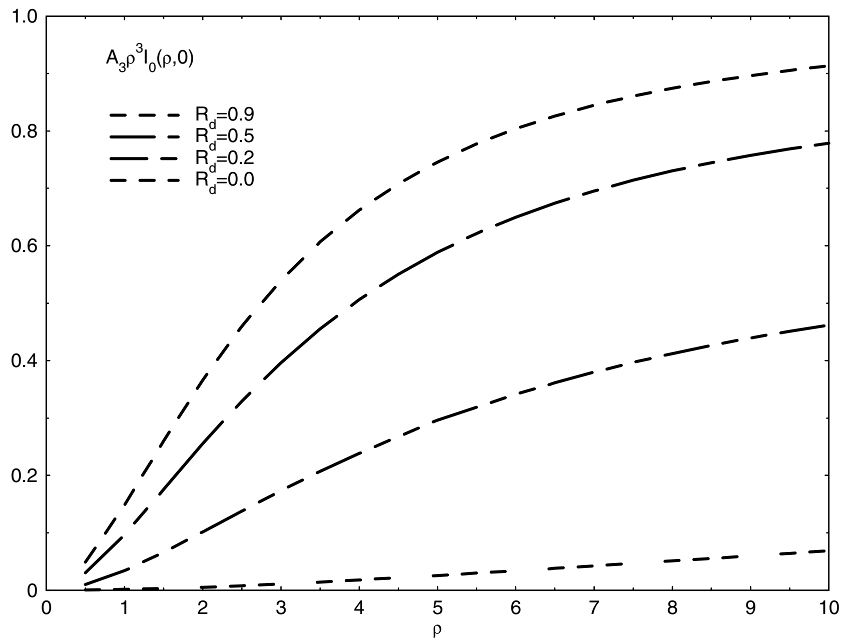

Figure 6. Normalized surface intensity for normal beam.

very small. All results in the figures are normalized such that, as $\rho \to \infty$, the quantity plotted tends to unity. Tables 4 and 5 show a range of values for the albedo $\alpha(\rho)$ as a function of ρ and the reflection coefficient R_d for $c = 1$. Figures 5 and 6 and tables 6 and 7 show

Table 4. Albedo $\alpha(\rho)$ for isotropic beam, $c = 1$, Lambert law of reflection.

ρ	$R_d = 0.0$	$R_d = 0.2$	$R_d = 0.5$	$R_d = 0.9$
0.5	0.1012	7.075(–2)	3.213(–2)	1.696(–3)
1.0	3.713(–2)	2.722(–2)	1.345(–2)	8.388(–4)
2.0	1.109(–2)	8.689(–3)	4.851(–3)	3.900(–4)
3.0	4.765(–3)	3.919(–3)	2.400(–3)	2.366(–4)
4.0	2.431(–3)	2.077(–3)	1.370(–3)	1.610(–4)
5.0	1.385(–3)	1.219(–3)	8.544(–4)	1.170(–4)
6.0	8.533(–4)	7.695(–4)	5.668(–4)	8.890(–5)
7.0	5.587(–4)	5.125(–4)	3.941(–4)	6.977(–5)
8.0	3.838(–4)	3.573(–4)	2.844(–4)	5.611(–5)
9.0	2.742(–4)	2.582(–4)	2.115(–4)	4.602(–5)
10.0	2.023(–4)	1.922(–4)	1.613(–4)	3.835(–5)

Table 5. Surface intensity $I_0(\rho, 0)$ for isotropic beam, $c = 1$, Lambert law of reflection.

ρ	$R_d = 0.0$	$R_d = 0.2$	$R_d = 0.5$	$R_d = 0.9$
0.5	0.6834	0.8843	1.2686	2.0953
1.0	0.2499	0.3392	0.5295	1.0330
2.0	7.196(–2)	0.1053	0.1874	0.4747
3.0	2.972(–2)	4.617(–2)	9.107(–2)	0.2856
4.0	1.467(–2)	2.392(–2)	5.132(–2)	0.1932
5.0	8.146(–3)	1.379(–2)	3.170(–2)	0.1401
6.0	4.926(–3)	8.591(–3)	2.089(–2)	0.1061
7.0	3.182(–3)	5.677(–3)	1.446(–2)	8.316(–2)
8.0	2.164(–3)	3.931(–3)	1.040(–2)	6.681(–2)
9.0	1.535(–3)	2.827(–3)	7.713(–3)	5.475(–2)
10.0	1.126(–3)	2.097(–3)	5.871(–3)	4.560(–2)

Table 6. Albedo $\alpha(\rho)$ for normal beam, $c = 1$, Lambert law of reflection.

ρ	$R_d = 0.0$	$R_d = 0.2$	$R_d = 0.5$	$R_d = 0.9$
0.5	8.991(–2)	6.296(–2)	2.869(–2)	1.526(–3)
1.0	3.326(–2)	2.443(–2)	1.212(–2)	7.631(–4)
2.0	1.047(–2)	8.187(–3)	4.557(–3)	3.663(–4)
3.0	4.743(–3)	3.866(–3)	2.337(–3)	2.274(–4)
4.0	2.528(–3)	2.126(–3)	1.372(–3)	1.570(–4)
5.0	1.491(–3)	1.284(–3)	8.736(–4)	1.153(–4)
6.0	9.449(–4)	8.294(–4)	5.892(–4)	8.828(–5)
7.0	6.324(–4)	5.635(–4)	4.150(–4)	6.966(–5)
8.0	4.422(–4)	3.986(–4)	3.026(–4)	5.626(–5)
9.0	3.203(–4)	2.915(–4)	2.270(–4)	4.629(–5)
10.0	2.390(–4)	2.193(–4)	1.743(–4)	3.867(–5)

similar quantities for a normal beam for the same ranges of ρ and R_d. Both albedo and surface intensity have a similar asymptotic behaviour. We note, however, that as the internal reflection coefficient R_d increases, the rate at which the asymptotic behaviour is approached decreases markedly. Indeed for $R_d = 0.9$, the asymptotic behaviour is not achieved until ρ is many hundreds of mean free paths. However, we may see from the tables that as the value of $R_d$approaches unity, the albedo and the surface intensity become less dependent on ρ. For example, in table 4, for $R_d = 0$, the ratio $\alpha(0.5)/\alpha(10.0) \approx 500$, whereas for $R_d = 0.9$, the

Table 7. Surface intensity $I_0(\rho, 0)$ for normal beam, $c = 1$, Lambert law of reflection.

ρ	$R_d = 0.0$	$R_d = 0.2$	$R_d = 0.5$	$R_d = 0.9$
0.5	0.1807	0.2381	0.3481	0.5875
1.0	6.880(–2)	9.448(–2)	0.1494	0.2967
2.0	2.118(–2)	3.113(–2)	5.557(–2)	0.1414
3.0	9.252(–3)	1.433(–2)	2.806(–2)	8.717(–2)
4.0	4.788(–3)	7.723(–3)	1.628(–2)	5.992(–2)
5.0	2.761(–3)	4.596(–3)	1.035(–2)	4.387(–2)
6.0	1.722(–3)	2.936(–3)	6.899(–3)	3.353(–2)
7.0	1.140(–3)	1.979(–3)	4.838(–3)	2.642(–2)
8.0	7.903(–4)	1.393(–3)	3.517(–3)	2.132(–2)
9.0	5.691(–4)	1.014(–3)	2.632(–3)	1.753(–2)
10.0	4.227(–4)	7.599(–4)	2.018(–3)	1.463(–2)

ratio is 44 and for $R_d = 0.999$ it is 17. So internal reflection allows the intensity more time to smooth itself out spatially; a not unexpected result physically.

6. Conclusions

We have shown how, by using the principle of superposition, the problem of internal reflection by any law, specular or diffuse or indeed a linear combination of these, can be used to formulate an equation for the emergent angular distribution of radiation from a surface (the directional emissive power) and the surface scalar intensity. Measures of the way in which the radiation intensity behaves when a beam of arbitrary direction impinges on the surface are obtained via the local albedo and the mean distance of travel over the surface. Asymptotic estimates are also obtained for the scalar intensity in inverse powers of distance for the non-absorbing case. We have also developed a very efficient numerical procedure for inverting a Fourier–Bessel transform using the Shanks summation formula. For the case of diffuse Lambert reflection, a complete solution is available for normal and isotropically incident beams. Numerical results for the case of the arbitrary beam, which involves a generalized H-function with complex argument, will be presented in a subsequent paper. It should be added that problems of the type described here can also be solved using the Monte Carlo method and by the finite-element method, as well as other essentially numerical approaches, in far greater generality. However, an analytic solution offers a useful benchmark against which to compare the convergence of these other methods.

Appendix A. Generalized *H*-functions

As we have shown in W1, the explicit form for $H(1/p)$ can be written as

$$H\left(\frac{1}{p}\right) = \frac{p + \sqrt{1 + k^2}}{p + \sqrt{\nu^2 + k^2}} \exp\left(\frac{1}{\pi}\int_0^1 \frac{\mathrm{d}w \tan^{-1}\Lambda(w)}{w\sqrt{1 + k^2 w^2}\left(wp + \sqrt{1 + k^2 w^2}\right)}\right) \tag{A.1}$$

where

$$\Lambda(w) = \frac{\pi c w/2}{1 - \frac{cw}{2}\log\left(\frac{1+w}{1-w}\right)}. \tag{A.2}$$

For numerical purposes, it is convenient to write this as

$$\tan^{-1}\Lambda(w) = \frac{\pi}{2} - \tan^{-1}\left[\frac{2}{c\pi w} - \frac{1}{\pi}\log\left(\frac{1+w}{1-w}\right)\right]. \tag{A.3}$$

Alternatively, using a different contour (Williams 1971) one can write

$$H\left(\frac{1}{p}\right) = \frac{p+\sqrt{1+k^2}}{p+\sqrt{\nu^2+k^2}}\exp\left(-\frac{p}{\pi}\int_0^\infty \frac{\mathrm{d}t}{t^2+p^2}\log\Omega(t,k)\right) \tag{A.4}$$

where

$$\Omega(t,k) = \frac{t^2+k^2+1}{t^2+k^2+\nu^2}\left(1 - c\frac{\tan^{-1}\sqrt{t^2+k^2}}{\sqrt{t^2+k^2}}\right) \tag{A.5}$$

and ν is the root of

$$1 - \frac{c}{2\nu}\log\left(\frac{1+\nu}{1-\nu}\right) = 0. \tag{A.6}$$

A few useful results from the above are

$$H(0) = 1 \tag{A.7}$$

$$\frac{1}{H(\infty)} = \left(1 - \frac{c}{k}\tan^{-1}k\right)^{1/2}. \tag{A.8}$$

For small values of k and $c = 1$, we may write

$$H(\bar{w}) = H_0(w)(1 - kw + O(k^2)) \tag{A.9}$$

where H_0 is the classic H-function.

A further useful expansion for $c = 1$, derived by Elliott (1952) and which we have verified, is

$$\log H(1/p) = \log\left[\frac{p+\sqrt{1+k^2}}{p+k}\right] - \frac{p}{\pi}\int_0^\infty \frac{\mathrm{d}t\log\Omega(t,0)}{t^2+p^2} + (1-z_0)\frac{k^2}{2p} + O(k^4) \tag{A.10}$$

where $z_0 = 0.710\,446\,09\ldots$. We may also write

$$\log\left[\frac{p+\sqrt{1+k^2}}{p+k}\right] = \log\left(1+\frac{1}{p}\right) - \frac{k}{p} + \frac{1}{2}\left(\frac{1}{p^2} + \frac{1}{1+p}\right)k^2 + O(k^3). \tag{A.11}$$

If we expand $H(\bar{\mu})$, $\bar{\mu} = \mu/(1+\mathrm{i}f)$, in powers of k_1 and k_2, we find from A4 that

$$H(\bar{\mu}) = H_0(\mu)\{1 + \mathrm{i}A(\mu)\{k_1\cos\varphi + k_2\sin\varphi\} + \cdots\} \tag{A.12}$$

where

$$A(\mu) = -\mu\sqrt{1-\mu^2}\left[\frac{1}{1+\mu} + \hat{\Omega}(\mu)\right] \tag{A.13}$$

where

$$\hat{\Omega}(\mu) = \frac{2}{\pi}\int_0^\infty \frac{\mathrm{d}t t^2 K(t)}{1+\mu^2 t^2} \tag{A.14}$$

with

$$K(t) = \frac{t^2\tan^{-1}t - 3(t - \tan^{-1}t)}{2t^2(1+t^2)(t-\tan^{-1}t)}. \tag{A.15}$$

Then

$$\begin{aligned}\frac{\mu_0 H(\bar{\mu}_0)H(\bar{\mu})}{\mu_0(1+\mathrm{i}f) + \mu(1+\mathrm{i}f_0)} = \frac{\mu_0 H(\mu_0)H(\mu)}{\mu_0+\mu}&[1 - \mathrm{i}Z(\mu,\mu_0)(k_1\cos\varphi_0 + k_2\sin\varphi_0) \\ &- \mathrm{i}Z(\mu_0,\mu)(k_1\cos\varphi + k_2\sin\varphi) + \cdots]\end{aligned} \tag{A.16}$$

where

$$Z(\mu,\mu_0) = \sqrt{1-\mu_0^2}\left[\frac{\mu_0}{1+\mu_0} + \frac{\mu}{\mu+\mu_0} + \mu_0\hat{\Omega}(\mu_0)\right]. \tag{A.17}$$

Appendix B. Integral equation for the generalized *H*-function

By using the Wiener–Hopf procedure, we can deduce integral equations which relate the generalized H-functions discussed in appendix A. For example if z and p are complex numbers, we have

$$\frac{1}{H\left(\frac{1}{z}\right)} - \frac{1}{H\left(\frac{1}{p}\right)} = \frac{c}{4\pi}(z-p)\int_0^{2\pi} \mathrm{d}\varphi \int_0^1 \mathrm{d}\mu \frac{\bar{\mu} H(\bar{\mu})}{(1+z\bar{\mu})\,(1+\mathrm{i}f+p\mu)} \tag{B.1}$$

where $\bar{\mu} = \mu/(1+\mathrm{i}f)$, with $f = \sqrt{1-\mu^2}(k_1\cos\varphi + k_2\sin\varphi)$. Equation (B.1) may be transformed to

$$\frac{1}{H(\bar{w})} - \frac{1}{H(\bar{w}_0)} = \frac{c}{2}(\bar{w}_0 - \bar{w})\int_0^1 \mathrm{d}w' \frac{\bar{w}' H(\bar{w}')}{(1+k^2 w'^2)(\bar{w}_0+\bar{w}')(\bar{w}+\bar{w}')} \tag{B.2}$$

where $\bar{w} = w/\sqrt{1+k^2w^2}$. Unfortunately, these equations do not help to simplify expressions such as equation (47) or (50) as would be the case if $k = 0$. These equations were derived by Williams (1982) but are given again here for completeness. Equation (B.2) has been verified numerically using (A.1) for $H(w)$ and by transformation agrees with a result of Crosbie and Linsenbardt (1978).

Appendix C. The principle of superposition

In order to obtain equation ((10) we used a principle of superposition. The approach is briefly described here. Consider the problem which we solved in W1, which was that of the intensity arising from a pencil beam on the surface. Suppose that we have a solution, $G(\mu, \varphi; \mu_0, \varphi_0)(\mu < 0,\ 0 < \varphi < 2\pi)$, for the pencil beam problem, where G is the solution of the searchlight problem for the following boundary condition,

$$G(\mu, \varphi; \mu_0, \varphi_0) = \delta(\mu - \mu_0)\delta(\varphi - \varphi_0), \qquad \mu > 0, \qquad 0 < \varphi < 2\pi. \tag{C.1}$$

We may regard $G(\mu, \varphi; \mu_0, \varphi_0)$ as the Green function and its form is given by equation (8) of the text. Now suppose we have a more general boundary condition of the form

$$\bar{I}(k_1, k_2, 0, \mu, \varphi) = \Phi(k_1, k_2, \mu, \varphi) + \delta(\mu - \mu_0)\delta(\varphi - \varphi_0), \qquad \mu > 0, \qquad 0 < \varphi < 2\pi. \tag{C.2}$$

Then we have, by superposition, or by using the Green function principle

$$\bar{I}(k_1, k_2, 0, -\mu, \varphi) = \int_0^1 \mathrm{d}\mu' \int_0^{2\pi} \mathrm{d}\varphi' G(-\mu, \varphi; \mu', \varphi') \times [\Phi(k_1, k_2, \mu', \varphi') + \delta(\mu' - \mu_0)\delta(\varphi' - \varphi_0)] \tag{C.3}$$

or more simply

$$\bar{I}(k_1, k_2, 0, -\mu, \varphi) = G(-\mu, \varphi; \mu_0, \varphi_0) + \int_0^1 \mathrm{d}\mu' \int_0^{2\pi} \mathrm{d}\varphi' G(-\mu, \varphi; \mu', \varphi')\Phi(k_1, k_2, \mu', \varphi') \tag{C.4}$$

which is equivalent to equation (10).

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